Uitwerkingen Tentamen Kwantumfysica 1
26 APRIL 2006

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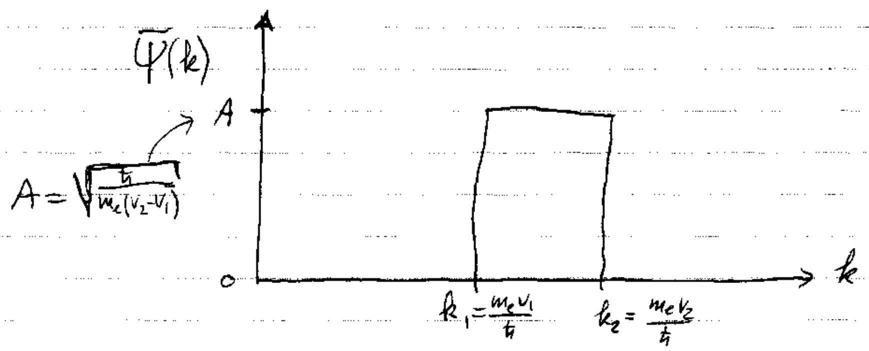
1a) Free particle ⇒ only kinetic energy, described with operator for momentum $\hat{p}_x$.

$$\hat{H} = \frac{\hat{p}_x^2}{2m_e}$$

1b) $k = \frac{p_x}{\hbar} = \frac{mv}{\hbar}$



$$\int_{-\infty}^{\infty} |\bar{\Psi}(k)|^2 dk = 1 \Rightarrow \int_{k_1}^{k_2} A^2 dk = 1 \Rightarrow$$

$$A^2(k_2 - k_1) = 1 \Rightarrow A = \sqrt{\frac{1}{(k_2-k_1)}} \Rightarrow$$

$$\boxed{A = \sqrt{\frac{\hbar}{m_e(v_2-v_1)}}}$$

~~voor Tijdsafhankelijk~~

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1c) $\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\Psi}(k)\, e^{ikx} dk$

$$= \frac{1}{\sqrt{2\pi}} \int_{k_1}^{k_2} A e^{ikx} dk = \frac{A}{\sqrt{2\pi}} \left[\frac{1}{ix} e^{ikx}\right]_{k_1}^{k_2}$$

$$= -\frac{iA}{\sqrt{2\pi}\, x}\left(e^{ik_2x} - e^{ik_1x}\right)$$

$$= -\frac{iA}{\sqrt{2\pi}\, x}\left(e^{i(k_c+\Delta k)x} - e^{i(k_c-\Delta k)x}\right)$$

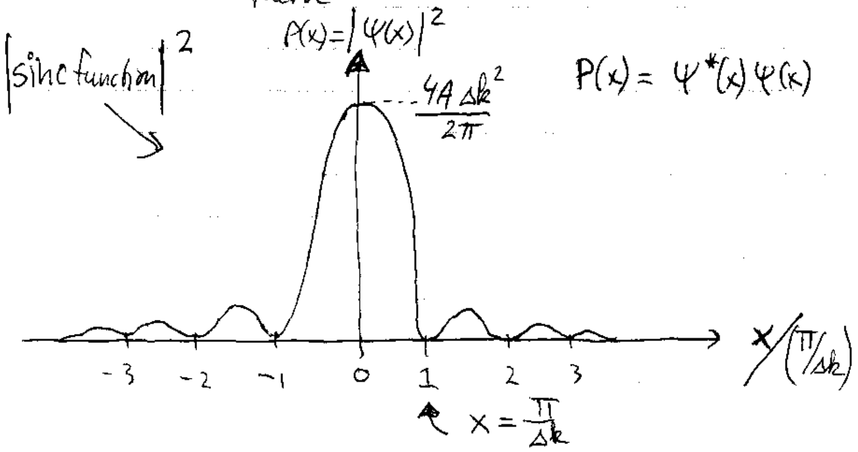
$$= -\frac{iA}{\sqrt{2\pi}\, x} e^{ik_cx}\left(e^{i\Delta kx} - e^{-i\Delta kx}\right)$$

$$= -\frac{iA}{\sqrt{2\pi}} \frac{1}{x} e^{ik_cx}\left(2i \sin(\Delta kx)\right) \Rightarrow$$

$$\Psi(x) = \underbrace{\left(\frac{2A\Delta k}{\sqrt{2\pi}}\right)}_{\text{Amplitude}} \underbrace{\left(e^{ik_cx}\right)}_{\text{phase factor}} \cdot \underbrace{\frac{\sin(\Delta kx)}{\Delta k x}}_{\text{sinc function}}$$

1d) $|\text{sinc function}|^2$

$P(x) = \Psi^*(x)\Psi(x)$



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1e) From the sketch of question b)

$$\Delta p_x \approx \frac{m_e(V_2-V_1)}{2}$$

From the sketch of question d)

$$\Delta x \approx \frac{2\pi}{\Delta k} = \frac{2\cdot 2\pi}{(k_2-k_1)} = \frac{4\pi\hbar}{m_e(V_2-V_1)}$$

$$\Delta p_x \cdot \Delta_x = \frac{4\pi\hbar}{m_e}\cdot\frac{m_e}{2} \approx 2\pi\hbar \Rightarrow$$

Is close to the Heisenberg limit

1f) Measurement of v collapses the state of an electron at a state with a very specific value of speed v_m (when the measurement result is v_m) $\Rightarrow$ As a function of k the state is close to a Dirac delta function centered at $k_m \Rightarrow$

$\bar{\Psi}(k) = \delta(k-k_m)$

$$k_m = \frac{m_e v_m}{\hbar} = \frac{9.1\cdot 10^{-31}\,kg \cdot 99.5\ m/s}{1.055\cdot 10^{-34}\ Js} = 8.6\cdot 10^5\,m^{-1}$$

1g) $\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{\Psi}(k)\, e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k-k_m)\,e^{ikx}\,dk \Rightarrow$

$\Psi(x) = \frac{1}{\sqrt{2\pi}}\, e^{i(k-k_m)x}$ $\Rightarrow$ a plane wave

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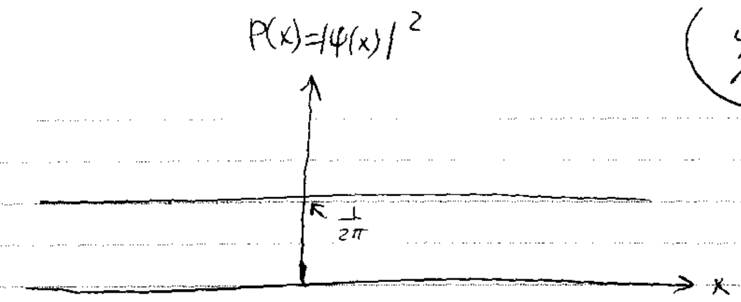
1h)



$$P(x) = |\Psi(x)|^2 = \Psi^*(x)\Psi(x) = \frac{1}{2\pi}$$

1i) The uncertainty in p_x (and thereby k) is very small, close to zero.
The uncertainty in position Δx then becomes close to infinite.
This can of course not happen in a real experimental setup, but it indicates what happens $\Rightarrow$ Δp_x very small and Δx very large.

1j) For an individual electron, the quantum uncertainty in the velocity is indeed very small. However, $\langle v\rangle$ will not always be exactly 100 m/s. Instead, it will have some value in the range $v_1 \dots v_2$. This value is prepared in a probabilistic manner, but one does know the value $\langle v\rangle = v_m$ from the measurement result. It will be different for each electron.
For an ensemble of electrons, nothing changes. The probability distribution $P(v)$ will still be as in the figure in the question.

2a) Time dependent

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

Time independent

$$H|\varphi_i\rangle = E_i|\varphi_i\rangle$$

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2b) $i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ (See also book p 85)

$\Rightarrow$ Time derivative equals constant operator times the function itself $\Rightarrow$ solutions in exponential format.

Assume solutions in the form

$$|\Psi\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\Psi_0\rangle$$, as they agree

with the Schrödinger equation $\Rightarrow$

State evolution of state $|\Psi_0\rangle$ can be described with operator $\hat{U}$.

2c) $\begin{cases} \hat{H}|\varphi_1\rangle = E_1|\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle = E_2|\varphi_2\rangle \end{cases} \Rightarrow \begin{array}{l} \langle\varphi_i|\hat{H}|\varphi_i\rangle = E_i \\ \langle\varphi_i \hat{H}|\varphi_j\rangle = 0 \\ \text{for } i \neq j \end{array}$

If $\hat{D}$ commutes with $\hat{H}$, then one has

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$\langle\varphi_1|\hat{D}|\varphi_1\rangle =$ constant, say D_1

$\langle\varphi_2|\hat{D}|\varphi_2\rangle =$ constant, say D_2

$\langle\varphi_i|\hat{D}|\varphi_j\rangle = 0$ for $i \neq j$, since

$\hat{D}$ then has the same eigenvectors as $\hat{H}$,

$|\varphi_i\rangle$ and $|\varphi_j\rangle$ are orthogonal, and

$\hat{D}|\varphi_i\rangle = D_i|\varphi_i\rangle \Rightarrow \langle\varphi_i|\hat{D}|\varphi_j\rangle = 0$

Say the arbitrary initial state at $t=0$ is $|\Psi_0\rangle = \alpha|\varphi_1\rangle + \beta|\varphi_2\rangle$. We then get for $\langle\hat{D}(t)\rangle$

$\langle\hat{D}(t)\rangle = \langle\Psi(t)|\hat{D}|\Psi(t)\rangle =$

$\langle\Psi_0|\hat{U}^+\hat{D}\hat{U}|\Psi_0\rangle =$

$$\left(e^{\frac{iE_1t}{\hbar}}\alpha^*\langle\varphi_1| + e^{\frac{iE_2t}{\hbar}}\beta^*\langle\varphi_2|\right)\hat{D}\left(e^{-\frac{iE_1t}{\hbar}}\alpha|\varphi_1\rangle + e^{-\frac{iE_2t}{\hbar}}\beta|\varphi_2\rangle\right).$$

With $\langle\varphi_1|\hat{D}|\varphi_2\rangle = \langle\varphi_2|\hat{D}|\varphi_1\rangle = 0$, this gives $\langle\hat{D}(t)\rangle = \alpha^*\alpha D_1 + \beta^*\beta D_2 \Rightarrow$

This is constant in time.

3a) $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$

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$\hat{H} = \frac{\hat{p}^2}{2m} + V_\infty$, with $V_\infty = \infty$ for region 1
$\hat{H} = \hat{p}^2/2m$ for region 2
$\hat{H} = \hat{p}^2/2m + V_0$ for region 3
$\hat{H} = \hat{p}^2/2m + V_\infty$ for region 4

3b) The energy eigen states are all bound states ⇒ these have a very particular value for the energy (the energy eigen value)

Since the potential energy is constant in each region 2, 3, the wave function must have a definite value for k^2 in each region (otherwise, there is no agreement with the time-independent Schrödinger equation: $-\frac{\hbar^2}{2m}\frac{\partial^2 \varphi_i}{\partial x^2} = \underbrace{(E-V)}_{\text{constant!}}\varphi_i$) ⇒

In each region 2, 3, each eigen state must therefore be in the form of a super position of 2 plane waves with the same value for $|k|$ (set by the kinetic energy $\frac{\hbar^2 k^2}{2m}$)

For region 1, 4, $V(x) = \infty$, so for all energy eigenstates with finite energy, there is zero probability for the electron to be in region 1 or 4.

3c) See book p 189.

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We have $E_g > V_0$ ⇒

$$k_2 = \sqrt{\frac{2m E_g}{\hbar^2}}$$

$$k_3 = \sqrt{\frac{2m(E_g - V_0)}{\hbar^2}}$$

Time-independent Schrödinger equation in x-representation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi_i(x) + V(x)\varphi_i(x) = E_i\varphi_i(x) \Rightarrow$$

$$\frac{\partial^2}{\partial x^2}\varphi_i(x) = \frac{2m(E_i - V)}{\hbar^2}\varphi_i(x) \Rightarrow$$

$$\frac{\partial^2}{\partial x^2}\varphi_i(x) = -k^2\varphi_i(x) \qquad (Eq. 1)$$

⇒ Has form that $\frac{\partial^2}{\partial x^2}$ of φ is equal to constant times φ itself ⇒ has solutions in exponential form $\varphi_i \propto e^{ikx}$ or e^{kx}

For $E_i > V$, the solutions are consistent with (Eq. 1) if one assume

$k = \sqrt{\frac{2m(E_i - V)}{\hbar^2}}$, with solutions in the form $\varphi_i \propto e^{\pm ikx}$.

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3d) 1° $\varphi_i(x)$ must be continuous at $x=0 \Rightarrow$

$$A+B=C+D \Rightarrow A+B-C-D=0$$

2° $\frac{\partial \varphi_i(x)}{dx}$ must be continuous at $x=0 \Rightarrow$

$$ik_2 A - ik_2 B = ik_3 C - ik_3 D \Rightarrow$$

$$ik_2 A - ik_2 B - ik_3 C + ik_3 D = 0$$

3° $\varphi_i(x)$ must be continuous at $x=-a$ (and a node of a standing wave). $\varphi_i(x)=0$ for $x<-a \Rightarrow \varphi_i(-a)\equiv 0 \Rightarrow$

$$Ae^{-ik_2 a} + Be^{ik_2 a} = 0$$

(Note, the derivative $\frac{\partial \varphi_i(x)}{\partial x}$ does not need to be continuous at $x=-a$, as V jumps to ∞)

4° Like 3°, for $x=a \Rightarrow$

$$Ce^{ik_3 a} + De^{-ik_3 a} = 0$$

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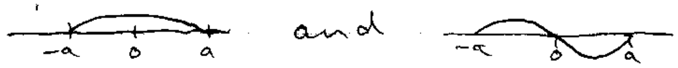
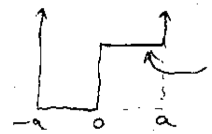
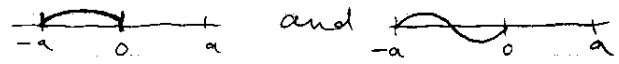
3e) The answer on 3d) shows 4 equations for solving A, B, C, D, that are linear in A, B, C, D. $\Rightarrow$ Can be solved with linear algebra techniques. Each row 1…4 of matrix M is one of the equations 1°, 2°, 3° and 4° of 3d)

3f) For this potential  the states are

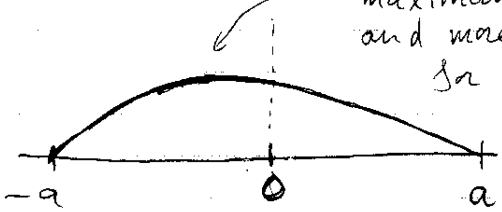


For this potential  ← V_0 very high

the states are 

$\Rightarrow$ For the system of problem 3

ground state



first excited state